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PERIODIC AND ALMOST PERIODIC SOLUTIONS OF STRONGLY NON-LINEAR IMPULSIVE SYSTEMS†

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The problem of the existence and stability of periodic and almost periodic solutions of strongly non-linear impulsive systems is investigated. The Poincaré method [1] is justified for the case of an isolated generating solution. A dynamical system consisting of a bead on a vibrating surface is considered as an example.

The small parameter method for investigating systems with discontinuous solutions was previously applied [2, 3] to the case when the periodic solution is non-isolated.

A method is used below for reducing the investigation of a system of equations with impulsive actions on surfaces to equations with fixed moments of inpulsive action.

1. BASIC DEFINITIONS

LET $G_{\mathbf{x}}$ BE a domain in R^n with compact closure and let $\mu_0>0$ be a fixed real number. On the set

$$G=\{(\mathbf{x},t,i,\mu)\,|\,\mathbf{x}\in G_{\mathbf{x}},\, t\in R,\, i=0,\pm 1,\ldots,-\mu_0<\mu<\mu_0\}$$

we consider a system of differential equations with impulsive actions on surfaces, the system having the form

$$\begin{aligned}
&d\mathbf{x}/dt=\mathbf{f}(t,\mathbf{x})+\mu\mathbf{g}(t,\mathbf{x},\mu),\quad t\neq t_i(\mathbf{x})+\mu\tau_i(\mathbf{x},\mu)\\
&\Delta\mathbf{x}|_{t=t_i(\mathbf{x})+\mu\tau_i(\mathbf{x},\mu)}=\mathbf{I}_i(\mathbf{x})+\mu\mathbf{W}_i(\mathbf{x},\mu)\\
&\mathbf{f}\in C^{(0,2)}(G)\cap C^{(1,2)}(G_0),\ \mathbf{g}\in C^{(0,1,1)}(G)\cap C^{(1,2,2)}(G_0)
\end{aligned}\tag{1.1}$$

where $\Delta\mathbf{x}|_{t=\theta}=\mathbf{x}(\theta+)=\mathbf{x}(\theta)$, I_i, W_i and τ_i are twice continuously differentiable functions, and G_0 is the union of some neighbourhoods of the surfaces $t=t_i(x)$.

The process defined by (1.1) for fixed μ operates as follows: the representational point $P_t=[t;\mathbf{x}(t)]$ leaves the point $(t_0,\mathbf{x}_0)$ and moves along the curve $t;\mathbf{x}(t)$ defined by the solution $\mathbf{x}(t)=\mathbf{x}(t,t_0,\mathbf{x}_0)$ of the equation

$$d\mathbf{x}/dt=f(t,\mathbf{x})+\mu\mathbf{g}(t,\mathbf{x},\mu)\tag{1.2}$$

The motion along this curve terminates at time $t=\theta_i$ when the point P_t arrives at one of the surfaces of discontinuity so that $\theta_i=t_i[\mathbf{x}(\theta_i)]+\mu\tau_i(\mathbf{x}(\theta_i),\mu)$.

At that moment the point P_t performs a jump $\Delta\mathbf{x}=\mathbf{I}_i[\mathbf{x}(\theta_i)]+\mu\mathbf{W}(\mathbf{X}(\theta_i),\mu)$ and then proceeds to move along the curve $t;\mathbf{x}(t)$ described by a solution $\mathbf{x}(t)=\mathbf{x}[t,\theta_i,\mathbf{x}(\theta_i+)]$ of system (1.2), etc. Thus the solution of Eq. (1.1) is a function that is piecewise-continuous, continuous on the left, and has discontinuities of the first kind. The basic results of the theory of differential equations with impulsive actions are given in [4].

The difficulty of investigating system (1.1) lies in the fact that the points of discontinuity of different solutions do not, in general, coincide. Hence, in order to better describe the asymptotic

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properties of the solutions and their dependence on the initial data and parameter, we introduce the following definitions.

Let $\mathbf{x}(t)$ be a solution of system (1.1) defined in an interval U. (U can be a finite section of a line, the whole line, or a half-line.)

We say that a solution $\mathbf{y}(t)$ of this system lies in an ϵ-neighbourhood of the solution $\mathbf{x}(t)$ if: (1) the measure of the symmetric difference of the domains of existence of these solutions is no greater than ϵ; (2) points of discontinuity of the solution $\mathbf{y}(t)$ lie in ϵ-neighbourhoods of points of discontinuity of $\mathbf{x}(t)$; (3) for all $t\in U$ lying inside ϵ-neighbourhoods of points of discontinuity of $\mathbf{x}(t)$ the inequality $\|\mathbf{x}(t)-\mathbf{y}(t)\|\leqslant\epsilon$ is satisfied.

The B-topology is the topology defined by ϵ-neighbourhoods of piecewise-continuous functions.

The solution $\mathbf{x}(t)$ is B-stable if $U=\{t\in R\,|\,t\geqslant t_0\}$, $t_0\in R$, and for any $\epsilon>0$ a real $\delta>0$ exists such that any solution $\mathbf{y}(t)$ satisfying the condition $\|\mathbf{x}(t_0)-\mathbf{y}(t_0)\|<\delta$ belongs to the ϵ-neighbourhood of $\mathbf{x}(t)$. [The point t_0 should not be a point of discontinuity of the solutions $\mathbf{x}(t)$ or $\mathbf{y}(t)$.]

A B-stable solution $\mathbf{x}(t)$ is called B-asymptotically stable if a $\delta>0$ exists such that for any real $\epsilon>0$ one can find a real number $\theta>t_0$ such that any solution $\mathbf{y}(t)$ satisfying the inequality $\|\mathbf{x}(t_0)-\mathbf{y}(t_0)\|<\delta$ belongs to the ϵ-neighbourhood of $\mathbf{x}(t)$ with $U=\{t\in R\,|\,t\geqslant\theta\}$.

Suppose $\mathbf{x}(t)=\mathbf{x}(t,\mu_0)$, $\mathbf{x}(t_0)=\mathbf{x}_0$ is a solution of system (1.1) and U is a bounded set. The solution $\mathbf{y}(t)=\mathbf{x}(t,\mu_0+\Delta\mu)$ also satisfies the initial condition $\mathbf{y}(t_0)=\mathbf{x}_0$, and θ_i, ξ_i are respectively the points of discontinuity of $\mathbf{x}(t)$ and $\mathbf{y}(t)$.

We say that a piecewise-continuous function $\mathbf{u}(t)$ is a B-derivative of the solution $\mathbf{x}(t)$ with respect to the parameter μ if for all $t\in U$ lying outside the intervals $(\theta_i,\xi_i]$ when $\theta_i\leqslant\xi_i$ or outside the intervals $(\xi_i,\theta_i]$ when $\xi_i<\theta_i$ we have the relation $\mathbf{y}(t)=\mathbf{x}(t)+\mathbf{u}(t)\Delta\mu+o(|\Delta\mu|)$ and, furthermore, there exists a sequence of real numbers r_i such that for each i we have $\xi_i=\theta_i+r_i\Delta\mu+o(|\Delta\mu|)$.

One can similarly define B-derivatives of a solution of system (1.1) with respect to initial conditions [5].

A piecewise-continuous function $\varphi(t)$ defined and uniformly bounded in the set R with discontinuities of the first kind at points of the sequence θ_i, $\theta_i\to\pm\infty$ as $i\to\pm\infty$, uniformly continuous in the collection of intervals (θ_i,θ_{i+1}), $i=0,\pm1,\ldots$, is called an almost-periodic (a.p.) function if for any real $\epsilon>0$ there exists a relatively dense set of almost-ϵ periods τ such that each function $\varphi(t+\tau)$ lies in an ϵ-neighbourhood of $\varphi(t)$.

We consider the generating system for Eq. (1.1)

$$d\mathbf{x}/dt=\mathbf{f}(t,\mathbf{x}),\quad t\neq t_i(\mathbf{x}),\quad \Delta\mathbf{x}|_{t=t_i(\mathbf{x})}=\mathbf{I}_i(\mathbf{x}) \tag{1.3}$$

We assume that a real number $\omega>0$ and an integer $p>0$ exist for which the equalities $\mathbf{f}(t+\omega,\mathbf{x})=\mathbf{f}(t,\mathbf{x})$, $t_{i+p}(\mathbf{x})=t_i(\mathbf{x})+\omega$, $\mathbf{I}_{i+p}(\mathbf{x})=\mathbf{I}_i(\mathbf{x})$ are satisfied uniformly in the domain G, and Eq. (1.3) has a solution $\mathbf{x}=\psi(t)$ with period ω and points of discontinuity $t=\theta_i$, $0<\theta_1<\theta_2<\ldots<\theta_p<\omega$.

Let Γ be some neighbourhood of the integral curve of the solution $\psi(t)$ in the set $G_{\mathbf{x}}\times R$

$$M=\sup_{\Gamma}\|\mathbf{f}\|,\quad C=\sup_{\Gamma}\left\|\frac{\partial t_i}{\partial\mathbf{x}}\right\|$$

and that

$$\min\inf_{\Gamma}(t_i(\mathbf{x})-t_i(\mathbf{x}+\mathbf{I}_i(\mathbf{x})))>0,\quad MC<1 \tag{1.4}$$

$$\min(\inf_{G_{\mathbf{x}}}t_i(\mathbf{x})-\sup_{G_{\mathbf{x}}}t_{i-1}(\mathbf{x}))=\gamma>0 \tag{1.5}$$

Below, $\mathbf{f}$, $\mathbf{x}$ and $\mathbf{I}$ and their derivatives will be taken to be column-vectors, while derivatives of the functions t_i are row-vectors. Products of vectors and matrices are the usual products of rectangular matrices. The values of the functions at the points $[\theta_i,\psi(\theta_i)]$ and $[\theta_i,\psi(\theta_i+)]$ are written without indicating the values of the arguments, the second case being distinguished by a superscript plus sign.

It follows [4] from conditions (1.4) that for sufficiently small $|\mu|$ there is no "beating" of the solutions of (1.2) against the discontinuity surface.

The system of equations in variations about the solution $\mathbf{x} = \psi(t)$ has the form [5]

$$\frac{du}{dt} = \mathbf{A}(t)\mathbf{u}, \quad t \neq \theta_i, \quad \Delta\mathbf{u}|_{t=\theta_i} = \mathbf{P}_i\mathbf{u}$$

$$\mathbf{A}(t) = \frac{\partial \mathbf{f}(t, \psi(t))}{\partial \mathbf{x}}, \quad \mathbf{P}_i = (\mathbf{f} - \mathbf{f}^+)\frac{\partial t_i}{\partial \mathbf{x}}\left(1 - \frac{\partial t_i}{\partial \mathbf{x}}\cdot \mathbf{f}\right)^{-1} + \tag{1.6}$$

$$+\frac{\partial \mathbf{I}_i}{\partial \mathbf{x}}\left(\mathbf{E} + \mathbf{f}\frac{\partial t_i}{\partial \mathbf{x}}\left(1 - \frac{\partial t_i}{\partial \mathbf{x}}\cdot \mathbf{f}\right)^{-1}\right)$$

where $\mathbf{E}$ is the unit $(n\times n)$ matrix.

We fix i. Suppose $\mathbf{x}_0(t)$ is a solution of system (1.2) with initial conditions $\mathbf{x}_0(\theta_i) = \mathbf{x}$, $t = \xi_i$ is the instant when this solution meets the surface $t = t_i(\mathbf{x}) + \mu\tau_i(\mathbf{x}, \mu)$, and $\mathbf{x}_1(t)$ is the solution of the Cauchy problem $\mathbf{x}_1(\xi_i) = \mathbf{x}_0(\xi_i) + I_i[\mathbf{x}_0(\xi_i)] + \mu W_i(\mathbf{x}_0(\xi_i), \mu)$ of system (1.2). Assuming the existence of the solutions $\mathbf{x}_0$ and $\mathbf{x}_1$ we define the map

$$\mathbf{J}_i(\mathbf{x}, \mu) = \int_{\theta_i}^{\xi_i} [\mathbf{f}(u, \mathbf{x}_0(u)) + \mu\mathbf{g}(u, \mathbf{x}_0(u), \mu)]\,du +$$

$$+ \mathbf{I}_i\left(\mathbf{x} + \int_{\theta_i}^{\xi_i} [\mathbf{f}(u, \mathbf{x}_0(u)) + \mu\mathbf{g}(u, \mathbf{x}_0(u), \mu)]\,du\right) +$$

$$+ \int_{\xi_i}^{\theta_i} [\mathbf{f}(u, \mathbf{x}_1(u)) + \mu\mathbf{g}(u, \mathbf{x}_1(u), \mu)]\,du$$

and construct a system of differential equations with impulsive actions at fixed instants of time, having the form

$$\frac{dy}{dt} = \mathbf{f}(t, \mathbf{y}) + \mu\mathbf{g}(t, \mathbf{y}, \mu), t \neq \theta_i$$

$$\Delta\mathbf{y}|_{t=\theta_i} = \mathbf{J}_i(\mathbf{y}, \mu) \tag{1.7}$$

One can verify that the following property is valid [6] A: if $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions of Eqs (1.1) and (1.7) respectively, with identical initial conditions and a common domain of existence, and $t = \xi_i$ are the points of discontinuity of $\mathbf{x}(t)$, then for each i we have the equality $\mathbf{x}(\theta_i) = \mathbf{y}(\theta_i)$ if $\theta_i \leqslant \xi_i$ or else $\mathbf{x}(\theta_i) = \mathbf{y}(\theta_i+)$ if $\xi_i < \theta_i$.

Thus, if we write $\mathbf{J}_i(\mathbf{x}, 0) = \mathbf{Q}_i(\mathbf{x})$, the solution $\psi(t)$ of system (1.3) is also a solution of the equation

$$d\mathbf{x}/dt = \mathbf{f}(t, \mathbf{x}), \quad t \neq \theta_i, \quad \Delta\mathbf{x}|_{t=\theta_i} = \mathbf{Q}_i(\mathbf{x}) \tag{1.8}$$

Theorems 1 and 2 proved in this paper are generalizations of the corresponding assertions in [1].

2. PERIODIC SOLUTIONS

We assume, in addition to the conditions of Sec. 1, that the equalities $\mathbf{g}(t+\omega, \mathbf{x}, \mu) = \mathbf{g}(t, \mathbf{x}, \mu)$, $\mathbf{W}_{i+p} = \mathbf{W}_i$ and $\tau_{i+p} = \tau_i$ hold uniformly in the domain G.

Theorem 1. Suppose systems (1.1) and (1.3) satisfy the above conditions and the multipliers of Eq. (1.6) are not equal to unity.

Then for sufficiently small $|\mu|$ system (1.1) has a unique ω-periodic solution which tends in the B-topology to the solution $\mathbf{x} = \psi(t)$ of the generating equation (1.3) as $\mu \to 0$. If in addition all the multipliers of system (1.6) are distributed inside the unit circle, the ω-periodic solution of system (1.1) is B-asymptotically stable.

Proof. Suppose $\mathbf{x}(t, \eta, \mu)$, $\mathbf{x}(0, \eta, \mu) = \eta$ is a solution of Eq. (1.1), and $\psi(t) = \mathbf{x}(t, \eta_0, 0)$ is an ω-periodic solution of system (1.3). For $\mathbf{x}(t, \eta, \mu)$ to be an ω-periodic solution it is necessary and sufficient that

$$\mathbf{D}(\eta, \mu) \equiv \mathbf{x}(\omega, \eta, \mu) - \eta = 0 \tag{2.1}$$

is solvable for η.

Without loss of generality one can assume that the point $(\eta_0, 0)$, together with a neighbourhood, does not belong to any of the surfaces $t = t_i(\mathbf{x})$. From this and from the differentiability of the functions $\mathbf{f}$, $\mathbf{g}$, $\mathbf{I}$, $\mathbf{W}$, t_i, τ_i, it follows by the theorem on the existence of B-derivatives of solutions of impulsive systems with respect to intial data [5] that the Jacobian $\mathbf{D}'_\eta(\eta, \mu)$ exists and is continuous in a neighbourhood of the point $(\eta_0, 0)$.

Suppose now that $\mathbf{X}(\omega)$ is the monodromy matrix of system (1.6). By definition, the variational system of equations $\mathbf{D}'_\eta(\eta_0, 0) = \det(\mathbf{X}(\omega) - \mathbf{E})$, and consequently, by virtue of the assumption on the multipliers, $\mathbf{D}'_\eta(\eta_0, 0) \neq 0$. From this follows the existence of a unique ω-periodic solution $\mathbf{x}(t, \eta, \mu)$. Its $\psi(t)$ limit in the B-topology follows from the theorem of the continuous dependence of solutions of impulsive systems on the initial data and parameters [7].

Suppose now that all multipliers of system (1.6) have a modulus smaller than one. By the assumptions of the theorem there exist B-derivatives of the solution $\mathbf{x}(t, \eta, \mu)$ with respect to the initial data η_j ($j = 1, 2, \ldots, n$) which form a normalized fundamental matrix of solutions corresponding to the $\mathbf{x}(t, \eta, \mu)$ system of equations in variations. That fundamental matrix of solutions at the point $t = \omega$ is the monodromy matrix. Since B-derivatives depend continuously on μ, the corresponding multipliers will, for sufficiently small $|\mu|$, have a modulus smaller than one. Consequently, by the generalization of the Lyapunov–Poincaré theorem [5] the ω-periodic solution of system (1.1) will be B-asymptotically stable. The theorem is proved.

3. ALMOST PERIODIC SOLUTIONS

Suppose that the conditions of Sec. 1 hold for (1.1), and moreover that the function $\mathbf{g}$ is Bohr a.p. with respect to t, while the sequences $\mathbf{W}_i$, τ_i are uniformly a.p. in the domain G. Suppose also that the parameter μ, in addition to the previous smallness requirements, satisfies the inequality

$$|\mu| \sup_G |\tau_i| < \gamma/2.$$

On the basis of these assumptions and the similarity to the proof of Lemma 5 in [6] one can verify that the sequence $\mathbf{J}_i$ is a.p. uniformly with respect to $\mathbf{x}$ and μ. Furthermore, by Lemma 1.5 the functions $\mathbf{J}_i$ have continuous second-order derivatives with respect to the variables x_j, $j = \overline{1, n}$ and μ.† Applying the Hadamard lemma, we find that the representation $\mathbf{J}_i(\mathbf{x}, \mu) = \mathbf{Q}_i(\mathbf{x}) = \mu\mathbf{H}_i(\mathbf{x}, \mu)$ holds for each i, where $\mathbf{H}_i$ is a continuously differentiable function, $\mathbf{Q}_i$ is a twice continuously differentiable function and $\mathbf{H}_i$, $\mathbf{Q}_i$ are a.p. sequences. One can also verify that $\partial\mathbf{Q}_i[\psi(\theta_i)]/\partial\mathbf{x} = \mathbf{P}_i$. Thus, in some neighbourhood of the integral curve of the solution $\mathbf{x} = \psi(t)$ in G, (1.7) can be written in the form

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{f}(t, \mathbf{x}) + \mu\mathbf{g}(t, \mathbf{x}, \mu), \quad t \neq \theta_i \\ \Delta\mathbf{x}|_{t=\theta_i} &= \mathbf{Q}_i(\mathbf{x}) + \mu\mathbf{H}_i(\mathbf{x}, \mu). \end{aligned} \tag{3.1}$$

†SAMOILENKO A. M., PERESTYUK N. A. and AKHMETOV M. U., Differential properties of solutions and integral surfaces of non-linear impulsive systems. Preprint No. 37, Institute of Mathematics, Kiev, 1990.

Performing the replacement $\mathbf{x} = \psi(t) + \mathbf{z}$ in (3.1), we go over to the system

$$\begin{aligned} \frac{d\mathbf{z}}{dt} &= \mathbf{A}(t)\mathbf{z} + \Psi(t, \mathbf{z}) + \mu \mathbf{g}(t, \psi(t), 0) + \mu \mathbf{F}(t, \mathbf{z}, \mu), \ t \neq \theta_i \\ \Delta \mathbf{z}|_{t=\theta_i} &= \mathbf{P}_i \mathbf{z} + \mathbf{S}_i(\mathbf{z}) + \mu \mathbf{H}_i(\psi(\theta_i), 0) + \mu \mathbf{V}_i(\mathbf{z}, \mu) \end{aligned} \tag{3.2}$$

in which

$$\begin{aligned} \Psi(t, \mathbf{z}) &= \mathbf{f}(t, \psi(t) + \mathbf{z}) - \mathbf{f}(t, \mathbf{z}) - \mathbf{A}(t)\mathbf{z} \\ \mathbf{F}(t, \mathbf{z}\ \mu) &= \mathbf{g}(t, \psi(t) + \mathbf{z}, \mu) - \mathbf{g}(t, \psi(t), 0) \\ \mathbf{S}_i(\mathbf{z}) &= \mathbf{Q}_i(\psi(\theta_i) + \mathbf{z}) - \mathbf{Q}_i(\psi(\theta_i)) - \mathbf{P}_i \mathbf{z} \\ \mathbf{V}_i(\mathbf{z}, \mu) &= \mathbf{H}_i(\psi(\theta_i) + \mathbf{z}, \mu) - \mathbf{H}_i(\psi(\theta_i), 0) \end{aligned}$$

One can verify that the functions Ψ, **F**, **S** and **V** satisfy the inequalities

$$\begin{aligned} &\|\Psi(t, \mathbf{z})\| + \|\mathbf{S}_i(\mathbf{z})\| \leqslant m \|\mathbf{z}\|^2 \\ &\|\Psi(t, \mathbf{z}_1) - \Psi(t, \mathbf{z}_2)\| + \|\mathbf{S}_i(\mathbf{z}_1) - \mathbf{S}_i(\mathbf{z}_2)\| \leqslant \\ &\leqslant l\,(\|\mathbf{z}_1\| + \|\mathbf{z}_2\|)\|\mathbf{z}_1 - \mathbf{z}_2\| \\ &\|\mathbf{F}(t, \mathbf{z}_1, \mu_1) - \mathbf{F}(t, \mathbf{z}_2, \mu_2)\| + \|\mathbf{V}_i(\mathbf{z}_1, \mu_1) - \\ &- \mathbf{V}_i(\mathbf{z}_2, \mu_2)\| \leqslant k(\|\mathbf{z}_1 - \mathbf{z}_2\| + |\mu_1 - \mu_2|) \end{aligned}$$

for all i and $\mathbf{z}$, where m and k are non-negative constants, and the function $l(u) \to 0$ as $u \to 0$. We consider the equation

$$\begin{aligned} d\mathbf{z}/dt &= \mathbf{A}(t)\mathbf{z} + \mu \mathbf{g}(t, \psi(t), 0), \ t \neq \theta_i \\ \Delta \mathbf{z}|_{t=\theta_i} &= \mathbf{P}_i \mathbf{z} + \mu \mathbf{H}_i(\psi(\theta_i), 0) \end{aligned} \tag{3.3}$$

If one assumes that the multipliers of (1.6) do not lie on the unit circle, then [4] a Green's function $\Omega(t, s)$ for (3.3) exists, for which constants $K \geqslant 1$ and $\alpha > 0$ exist such that

$$\|\Omega(t, s)\| \leqslant K \exp(-\alpha|t - s|), \quad -\infty < t, \ s < +\infty$$

We will introduce the following notation

$$\begin{aligned} L &= \sup_t \|\mathbf{g}(t, \psi(t), 0)\| + \sup_i \|\mathbf{H}_i(\psi(\theta_i), 0)\| \\ K_0 &= K \max\left(\frac{2}{\alpha}, \frac{2}{1 - e^{-\alpha\gamma}}\right) \end{aligned}$$

Because $g[t, \psi(t), 0]$ is an a.p. function, and $\mathbf{H}_i[\psi(\theta_i), 0]$ is an a.p. sequence, the function

$$\varphi_0(t) = \mu \int_{\infty}^{\infty} \Omega(t, u) \mathbf{g}(u, \psi(u), 0)\, du + \mu \sum_{i=-\infty}^{\infty} \Omega(t, \theta_i) \mathbf{H}_i(\psi(\theta_i), 0)$$

is the unique a.p. solution of Eq. (3.3) and satisfies the inequality $\|\varphi_0(t)\| \leqslant |\mu| K_0 L$ [4].

The following theorem holds.

Theorem 2. If system (1.1) satisfies the above conditions, then for sufficiently small $|\mu|$ it has a unique a.p. solution $\zeta(t)$ which tends in the B-topology to the ω-periodic solution $\mathbf{x} = \psi(t)$ of the generating equation (1.3) as $\mu \to 0$.

Proof. We fix the positive N and construct the set Π of all discontinuous a.p. functions $\varphi(t)$ that have discontinuities at points of the sequence θ_i and satisfy the inequality $\|\varphi(t) - \varphi_0(t)\| \leqslant |\mu| N$ for $t \in R$. We define in this space the norm

$$\| \varphi \|_0 = \sup_t \| \varphi(t) \|$$

Suppose that the operator Φ acts in Π

$$\Phi(\varphi(t)) = \varphi_0(t) +$$
$$+ \int_{-\infty}^{\infty} \Omega(t, u) [\Psi(u, \varphi(u)) + \mu \mathbf{F}(u, \varphi(u), \mu)] du +$$
$$+ \sum_{i=-\infty}^{\infty} \Omega(t, \theta_i) [\mathbf{S}_i(\varphi(\theta_i)) + \mu \mathbf{V}_i(\varphi(\theta_i), \mu)]$$

We will verify that for sufficiently small $|\mu|$ the relation $\Phi(\varphi) \in \Pi$ holds for each $\varphi \in \Pi$. Firstly, we have

$$\| \Phi(\varphi(t)) - \varphi_0(t) \| \leqslant \int_{-\infty}^{\infty} K e^{-\alpha|t-u|} [\mu^2 m (N + K_0 L)^2 +$$
$$+ \mu^2 k (N + K_0 + 1)] du + \sum_{i=-\infty}^{\infty} K e^{-\alpha|t-\theta_i|} [\mu^2 m (N + K_0 L)^2 +$$
$$+ \mu^2 k (N + K_0 + 1)] \leqslant \mu^2 K_0 [m (N + K_0 L)^2 + (N + K_0 + 1)]$$

and hence, if we assume that the condition

$$|\mu| < N [K_0 (m (N + K_0 L)^2 + (N + K_0 + 1))]^{-1}$$

holds, the inequality $\|\Phi(\varphi(t)) - \varphi_0(t)\| \leqslant |\mu| N$ is satisfied.

Then from the almost periodicity of the functions $\Psi(t, \varphi(t))$, $\mathbf{F}(t, \varphi(t), \mu)$ and sequences $\mathbf{S}_i(\varphi(\theta_i))$, $\mathbf{V}_i(\varphi(\theta_i), \mu)$, and using Lemma 24.4 of [4], we find that the function $\Phi(\varphi(t))$ is almost periodic. Suppose now that $\varphi_1, \varphi_2 \in \Pi$. For these functions we have

$$\| \Phi(\varphi_1) - \Phi(\varphi_2) \| \leqslant$$
$$\leqslant \int_{-\infty}^{\infty} K e^{-\alpha|t-u|} [l(2|\mu| N) \| \varphi_1 - \varphi_2 \|_0 + |\mu| k \| \varphi_1 - \varphi_2 \|_0] du +$$
$$+ \sum_{i=-\infty}^{\infty} K e^{-\alpha|t-\theta_i|} [l(2|\mu| N) \| \varphi_1 - \varphi_2 \|_0 +$$
$$+ |\mu| k \| \varphi_1 - \varphi_2 \|_0] \leqslant K_0 (l(2|\mu| N) + |\mu| k) \| \varphi_1 - \varphi_2 \|_0$$

From this, with the condition $K_0(l(2|\mu|N) + |\mu| k) < 1$, it follows that Φ is a contraction operator in Π. Thus, if $|\mu|$ is sufficiently small, the sequence of a.p. functions φ_k, where φ_0 is the solution of (3.3) and $\varphi_{k+1} = \Phi(\varphi_k)$ for $k = 0, 1, 2, \ldots$ converges to the a.p. function $\mathbf{v}(t)$ which is the solution of the equation $\mathbf{z} = \Phi(\mathbf{z})$.

Differentiating the expression $\mathbf{v} = \Phi(\mathbf{v})$ at points $t \neq \theta_i$ and checking that the discontinuity conditions are satisfied, we verify that $\mathbf{v}$ is the solution of system (3.2). Then the function $\eta = \psi(t) + \mathbf{v}(t)$ is the a.p. solution of Eq. (3.1) and simultaneously of system (1.7). It satisfies the inequality

$$\| \eta(t) - \psi(t) \| \leqslant |\mu| (K_0 L + N) \tag{3.4}$$

By property A we find that for sufficiently small $|\mu|$ system (1.1) has an a.p. solution $\zeta(t)$.

We will now prove the uniqueness of the a.p. solution $\zeta(t)$. Again using property A and the replacement $\mathbf{x} = \psi(t) + \mathbf{z}$ we find that the proof reduces to the verification of the uniqueness of the a.p. solution of system (3.2). We will assume the opposite, namely, that ψ_1 and ψ_2 are different a.p. solutions of (3.2). From the properties of the Green's function $\Omega(t, s)$ it follows that for $j = 1, 2$ we have the equality

$$\psi_j = \int_{-\infty}^{\infty} \Omega(t, u) [\Psi(u, \psi_j) + \mu \mathbf{F}(u, \psi_j, \mu)] du +$$

$$+ \sum_{-\infty}^{\infty} \Omega(t, \theta_i) [\mathbf{S}_i(\psi(\theta_i)) + \mu \mathbf{V}_i(\psi(\theta_i), \mu)]$$

From this there follows the relation

$$\| \psi_1 - \psi_2 \| \leqslant K_0 \ (l(2|\mu|N) + |\mu| k) \ \| \psi_1 - \psi_2 \|_0, t \in R.$$

But this equality is only possible provided $\|\psi_1 - \psi_2\|_0 = 0$, i.e. $\psi_1(t) \equiv \psi_2(t)$.

Finally, from inequality (3.4) and the continuous dependence of the solutions of system (1.1) on the initial data and the parameter, there follows the convergence $\zeta(t) \to \psi(t)$ in the B-topology as $\mu \to 0$.

The theorem is proved.

4. EXAMPLE

Consider a dynamical system consisting of a bead bouncing on a platform. Such a system has been previously investigated in [8]. We assume that the platform does not react to collisions with the bead and moves according to the law $X = X_0 \sin \omega t$. The motion of the bead between collisions is given by the formula

$$x = -g(t - \varphi)^2/2 + x_0^\bullet (t - \varphi) + x_0$$

where x_0 and $x_0^\bullet$ are the values of the coordinate and velocity of the bead at the instant $t = \varphi$ immediately after collision.

The dynamical system under investigation with the condition

$$\omega^2 > \frac{\pi g}{X_0} \frac{1-R}{1+R} \tag{4.1}$$

admits of a motion $x = \psi(t)$ with period $T = 2\pi/\omega$, which at the instant $t = \varphi$ given by the relation

$$\cos \varphi = \frac{\pi g}{X_0 \omega^2} \frac{1-R}{1+R}$$

(R being the coefficient of restitution) has the initial value

$$x_0 = X_0 \sqrt{1 - \cos^2 \varphi}, \ x_0^\bullet = \pi g/\omega$$

If we write $x = x_1$, and $dx/dt = x_2 \arcsin(x_1/X_0)\omega = t_0(x_1)$, then for the dynamical system under consideration one can construct a suitable mathematical model in the form of the following non-linear system of differential equations with impulsive actions

$$\begin{aligned} & dx_1/dt = x_2, \ dx_2/dt = -g, \ t \neq t_0(x_1) \\ & \Delta x_2 |_{t=t_0(x_1)} = (1+R)[X_0 \omega \cos(\arcsin \frac{x_1}{X_0}) - x_2] \end{aligned} \tag{4.2}$$

Because of the T-periodicity, the model is constructed only for the interval $[0, T]$. In order to consider the entire range $+\infty < t < +\infty$ it is sufficient to replace the surface $t = t_0(x_1)$ by the surfaces $t = t_i \equiv t_0(x_1) + 2\pi i/\omega$ in system (4.2).

It is obvious that (4.2) is a highly idealized model. It takes no account of drag in the medium, the unavoidable perturbations of the platform, and possible elastic couplings. Hence it is natural to consider a system of the form

$$\begin{aligned} & dx_1/dt = x_2 \\ & dx_2/dt = -g + \mu f(t, \mathbf{x}, \mu), \ t \neq t_i(\mathbf{x}) + \mu \tau_i(\mathbf{x}, \mu) \\ & \Delta x_2 |_{t=t_i(x) + \mu\tau_i(x, \mu)} = (1+R)[X_0 \omega \cos(\arcsin \frac{x_1}{X_0}) - x_2] + \mu I_i(\mathbf{x}, \mu) \end{aligned} \tag{4.3}$$

in which $\mathbf{x} = (x_1, x_2)$ and μ is a small parameter.

We shall assume that in (4.3) the functions f, I and τ have continuous second-order partial derivatives with respect to x_1, x_2 and μ, and that the function f is continuously differentiable with respect to t. System (4.2) is the generator of Eqs (4.3). Hence we begin the investigation with the former. The system of equations in variations about the solution $x = \psi(t)$ has the form

$$\begin{aligned} &du_1/dt = u_2,\ du_2/dt = 0,\ t \neq \varphi \\ &\Delta u_1|_{t=\varphi} = -(1+R)u_1 \\ &\Delta u_2|_{t=\varphi} = \frac{\omega}{2}\left[\frac{(1+R)^2}{\pi} - b(1-R)^2\right)\mu_1 - (1+R)u_2 \\ &b = \sqrt{\cos^{-2}\varphi - 1} \end{aligned} \tag{4.4}$$

The characteristic equation for (4.4) is

$$\rho^2 + (\pi b(1-R^2) - (1+R^2))\rho + R^2 = 0$$

We find from this equation that (4.4) does not have a unique multiplier provided

$$R = 1 \ \text{or}\ \omega^2 \neq \frac{\pi g}{X_0}\frac{1-R}{1+R} \tag{4.5}$$

A necessary and sufficient condition for the multipliers to be situated inside the unit sphere is the inequality

$$\frac{\pi g}{X_0}\frac{1-R}{1+R} < \omega^2 < \left\{\left[\frac{\pi g(1-R)}{X_0(1+R)}\right]^2 + \left[\frac{2g(1+R^2)}{X_0(1+R)^2}\right]^2\right\}^{1/2} \tag{4.6}$$

Relation (4.6) is identical to the condition obtained in [8] by the matching method.

We change to system (4.3). Assuming that $R \neq 1$, we consider two cases.

1. Suppose a function f with period T with respect to t and uniform with respect to $\mathbf{x}$, μ, $i = 0, \pm 1$, satisfies the equalities $I_{i+1} = I_i$, $\tau_{i+1} = \tau_i$. Then from relations (4.1), (4.5) and (4.6), we find according to Theorem 1 that if inequality (4.6) holds, system (4.3) admits of a unique B-asymptotically stable solution with period T for sufficiently small μ, which when $\mu \to 0$ tends in the B-topology to the solution $x = \psi(t)$ of system (4.2).

2. Suppose the function f is a.p. with respect to t in the Bohr sense and the sequences I_i and τ_i are a.p. uniformly with respect to $\mathbf{x}$ and μ. One can verify that if condition (4.5) is satisfied, the multipliers of Eqs (4.4) do not lie on the unit circle. Consequently, on the basis of Theorem 2, we conclude that when the inequality

$$\omega^2 > \frac{\pi g}{X_0}\frac{1-R}{1+R}$$

is satisfied, system (4.3) has a unique a.p. solution which tends in the B-topology to the solution $x = \psi(t)$ of Eqs (4.2) as $\mu \to 0$.

REFERENCES

1. MALKIN I. G., *Some Problems in the Theory of Non-linear Oscillations*. Gostekhizdat, Moscow, 1956.
2. KOLOVSKII M. Z., *The Use of the Small Parameter Method to Determine Discontinuous Periodic Solutions*. Izd. Leningrad. Politekhn. Inst., Leningrad, 1961.
3. NAGAYEV R. F., Periodic solutions of piecewise-continuous systems with a small parameter. *Prikl. Mat. Mekh.* **36**, 1059–1069, 1972.
4. SAMOILENKO A. M. and PERESTYUK N. A., *Differential Equations with Impulsive Actions*. Vishch. Shkola, Kiev, 1987.
5. AKHMETOV M. U. and PERESTYUK N. A., On the differential dependence of solutions of impulsive systems on the initial data. *Ukr. Mat. Zh.* **41**, 1028–1033, 1989.
6. AKHMETOV M. U. and PERESTYUK N. A., On the comparison method for differential equations with impulsive actions. *Diff. Equations* **26**, 1475–1483, 1990.

7. AKHMETOV M. U. and PERESTYUK N. A., Periodic solutions of systems of differential equations with impulsive actions on surfaces. In *Applications of Asymptotic Methods in the Theory of Non-linear Differential Equations*, pp. 11–17. Kiev, 1987.
8. KOBRINSKII A. Ye. and KOBRINSKII A. A., *Vibro-shock Systems*. Nauka, Moscow, 1973.

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